A Simple Non-Parametric Approach to Bond Futures Option Pricing

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Options on the Chicago Board of Trade's Treasury bond futures contracts were introduced in October 1982, and have been widely used since. Numerous valuation models for fixed-income options like these have been proposed but none has achieved widespread acceptance among practitioners. After surveying the academic literature, Hull [1993, p. 409] opines that "Interest rate options are more difficult to value than stock options, currency options, index options, and most futures options." This is true despite the increasing sophistication of interest rate option valuation models.

Practitioners have devised their own methods for predicting option prices. A study of stock index option valuation models concludes that some common, simple predictive schemes, based on two-week-ahead extrapolation of Black-Scholes model implied volatilities, outperform several academic models (see Jackwerth and Rubinstein [1996]).

This is not surprising, given the persistent pattern of "moneyness bias." That is, implied volatilities are inversely related to the exercise price (in other words, in-the-money calls (or out-of-the-money puts) tend to be underpriced relative to the other degrees of moneyness) that plagues parametric valuation models of stock index options, as documented by Bates [1996], Bakshi, Cao, and Chen [1997], and countless practitioners.

In what is perhaps the only systematic study of parametric valuation model performance in pricing CBOT note and bond futures options, Cakici, Chatterjee, and Wolf [1993] examine the Black [1976] futures options valuation model pricing performance during 1987, and conclude that:

> The in-the-money calls are underpriced by the model. This underpricing is most pronounced for the intermediate maturity of \$6 < T < 12, where T is maturity in weeks. These two results are comparable to Whaley [1986]…. As for the model's maturity bias, the underpricing of the short maturity in-the-money calls seems to be the only economically significant mispricing [1993, p. 7].

Examination of the pricing error classification of Hull [1993, pp. 436-438] indicates that in-the-money calls will be underpriced by the Black model when the actual risk-neutral distribution has a fatter left-hand tail than the lognormal does. This pattern of "moneyness bias," equivalently stated as the finding that implied volatilities are inversely related to the exercise price over some range and period of time, is the same as the pattern often found in stock index options.

Rubinstein [1994] conjectures that this implied volatility skew is due to market participants' fears of a serious stock market crash, causing the a fatter left-hand tail of the underlying stock index distribution. Supporting evidence for this is provided in Bates [1996] and in Corrado and Su [1997]. It is thus possible that the Black model mispricing of CBOT bond futures options is also due to fear of another bond market crash — perhaps on the order of the unprecedented high interest rates of the early 1980s, or the more recent bear market in 1994.

We develop a third way to predict option prices, which combines the risk-neutral valuation framework inherent in the formal option models with the flexibility of practitioner methods. First developed in Stutzer [1996], the *canonical* valuation model is a simplified, risk-neutral valuation method that permits the user to specify an individual assessment of the distribution of the underlying security price at option expiration. Then, it uses this distribution to estimate risk-neutral probabilities needed to value the option, as the risklessly discounted, risk-neutral expected value of its payoff at expiration.

In Stutzer [1996] I use a simple histogram of past stock index price relatives (i.e., gross returns) that includes the Crash of 1987 to assess the distribution of future stock index values at option expiration. I demonstrate how the canonical model can avoid the pattern of moneyness bias that plagues the parametric models.

Here we show that the canonical model predicts that in the historically typical range of bond futures prices, the Black model implied volatility of in-the-money calls should indeed be somewhat higher than of other calls, and that this pattern will be more pronounced for shorter term options, consistent with empirical evidence. The canonical model also predicts that the implied volatilities should generally be much higher when the underlying futures price is near historic lows (i.e., in regimes of relatively high long-term bond rates), consistent with the empirical finding that the level and the volatility of interest rates are directly related. The common pattern of moneyness bias does not plague the canonical model, which consequently outperforms the Black model.

I. CANONICAL MODELING AND OPTION PRICING THEORY

Option pricing theory posits the existence of a stochastic process for the underlying price movements, which generates the actual probability distribution of the underlying price at option expiration. To value the option, the process is transformed via Girsanov's theorem (Dothan [1990, p. 209]) into a risk-neutral process that generates a risk-neutral probability distribution for the underlying price at expiration. A predicted option price is the risklessly discounted, expected value of the option's payoff at expiration. Because neither the functional form of the stochastic process nor its parameters are ever known with certainty, neither the actual nor the risk-neutral probabilities can be known with certainty.

To illustrate, consider the well-known Black [1976] model for European futures options, which Cakici, Chatterjee, and Wolf [1993] use in their comprehensive empirical study of CBOT note and bond futures options valuation.[1] Black assumes that the nature of the stochastic process governing the underlying futures price movements is such that the continuously compounded growth rate of the futures price until option expiration is normally distributed. Further, this distribution of the futures price growth rate does *not* vary with the current futures price. The risk-neutral transformation results in a change of its mean, but not its functional form.

The result of the expected value calculation for the predicted call option price *Call* can be written:

$$\text{Call} = e^{-rT}\,[FN(d_1) - XN(d_2)] \tag{1}$$

where

$$d_1 = [\log(F/X) + 0.5\sigma^2 T]/\sigma\sqrt{T}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

r denotes the continuously compounded riskless discount rate; T is the time to maturity in years; F is the current underlying futures price; N is the cumulative normal distribution function; X is the exercise price; and σ is the annualized underlying futures return volatility.

If there is misspecification of the underlying stochastic process and/or use of the wrong process parameters in it, the Black model may misestimate the actual probabilities of the underlying value at option expiration, and hence misestimate the transformed risk-neutral probabilities used to provide the Black model value in Equation (1). In other words, the Black model, as well as any other parametric model, determines implicit estimates of both actual and risk-neutral probabilities.

Canonical valuation by contrast allows the user to specify a particular assessment about the actual distribution of the underlying at expiration, which provides the basis for our estimate of the risk-neutral probabilities. As a result, the futures price growth rate until option expiration does not have to be normal, nor does it have to have the same distribution for each possible current futures price, as assumed in the Black model.

We use the history of bond futures prices to form a catalogue of histograms of futures price growth rates (actually, price relatives), indexed by the current futures price. In this way, non-normal distributions of the futures price growth rate, which varies with the current futures price, are determined, consistent with past movements of the underlying bond futures price.

After all, it is unrealistic to think that the one-month-ahead percentage changes in the futures price will have the same distribution when the futures price is 110 (a fifteen-year, 8% bond yielding 6.9%) that it has when the futures price is only 62 (bond yielding 14.2%). In the latter case, one might expect the volatility of the distribution to be higher, or, in the presence of mean reversion, one might expect the futures price to be more likely to rise than it would after starting from 110.[2]

II. ASSESSING THE DISTRIBUTION OF THE UNDERLYING FUTURES PRICE

We obtain the entire series of daily closing prices for the CBOT bond futures contracts through mid-1996. For an option with time T to maturity and current underlying futures price of F, the entire series is searched for futures prices ±\$8 from F. When one is found, the same contract's price at time T-ahead is recorded (if the contract had not yet expired). The ratio of the two prices is recorded as the price relative or gross *"return"* R_h^F. Each of these is multiplied by F to form H possible values of the T-ahead futures price, denoted $P_h(T, F) = F\,R_h^F$. Assigning equal probability to each produces a simple non-parametric estimate of the T-ahead probability distribution of the futures price, conditional on the current futures price F.[3]

Exhibit 1 lists the number H of observed N-month-ahead (twenty-one trading days per month) bond futures prices subsequent to observing a futures price ±\$8 from some specific values of F. Note that the bulk of the observations are from historical periods where the futures price was in the middle range (between \$86 and \$102), corresponding to fifteen-year, 8% coupon bond yields between 7.8% and 9.8%. The fewest observations are in the range of futures prices seen most recently, i.e., ±\$8 from 110.

Even the relative lack of data in this range does

EXHIBIT 1

Number of Price-Relative "Returns" R_h, ±\$8 from F

Price	Expiration (months)			
F	1	2	3	4
62	7,624	7,484	7,298	7,052
78	8,020	7,717	7,466	7,220
94	13,887	13,409	12,943	12,368
110	2,876	2,643	2,411	2,179

not detract from the relative out-of-sample pricing performance of the canonical model for options with six to twelve weeks to expiration — the range where Cakici, Chatterjee, and Wolf [1993] find the most serious Black model mispricing.

The N-month-ahead return distributions are illustrated in the Exhibits 2-5, which are histograms (normalized so that the area underneath them always equals 1) of the returns $R_h^F = P_h(T, F)/F$, from the starting futures prices F in Exhibit 1. Note that there is substantial variation in the shapes of the distributions. The two top panels of each graph illustrate the substantial skewness of the returns distribution when the current futures price is relatively low (i.e., bond yields are relatively high). That is, the probability of unusually large, positive increases in the futures price is higher when the starting futures price is relatively low. There is also a greater spread in the top panels than the lower panels, illustrating the higher volatility associated with the returns in periods of low futures prices (high bond yields). These properties are totally absent in the Black model.

III. TRANSFORMING ACTUAL PROBABILITIES INTO RISK-NEUTRAL PROBABILITIES

The second step in canonical valuation is to estimate risk-neutral probabilities satisfying a martingale constraint, i.e. the risk neutral expected value of the time T-ahead futures price must equal the current futures price.[4] The canonical risk-neutral probability distribution is the distribution satisfying this constraint that is closest (in the sense of relative entropy) to the time T-ahead distribution constructed as described. Because the historical T-ahead futures prices are assigned equal (subjective) probability, minimization of the relative entropy is equivalent to maximization of the *Shannon entropy*, making this method another application of the maxi-

EXHIBIT 2
Normalized Histogram for One-Month Return Data

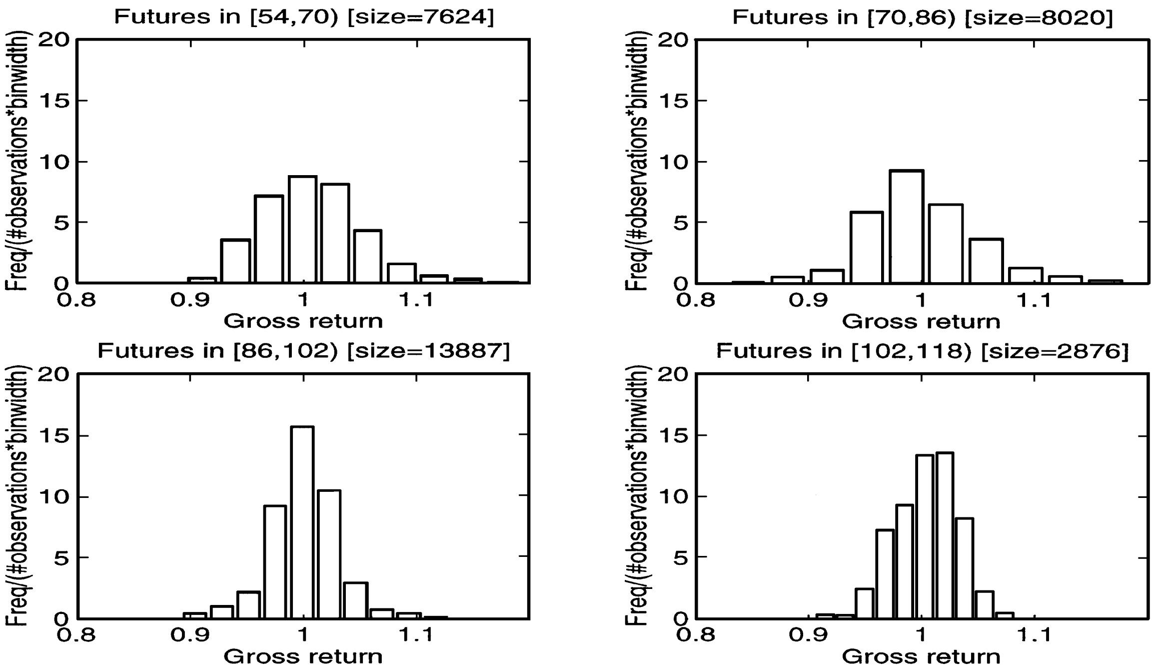


EXHIBIT 3
Normalized Histogram for Two Months' Return Data

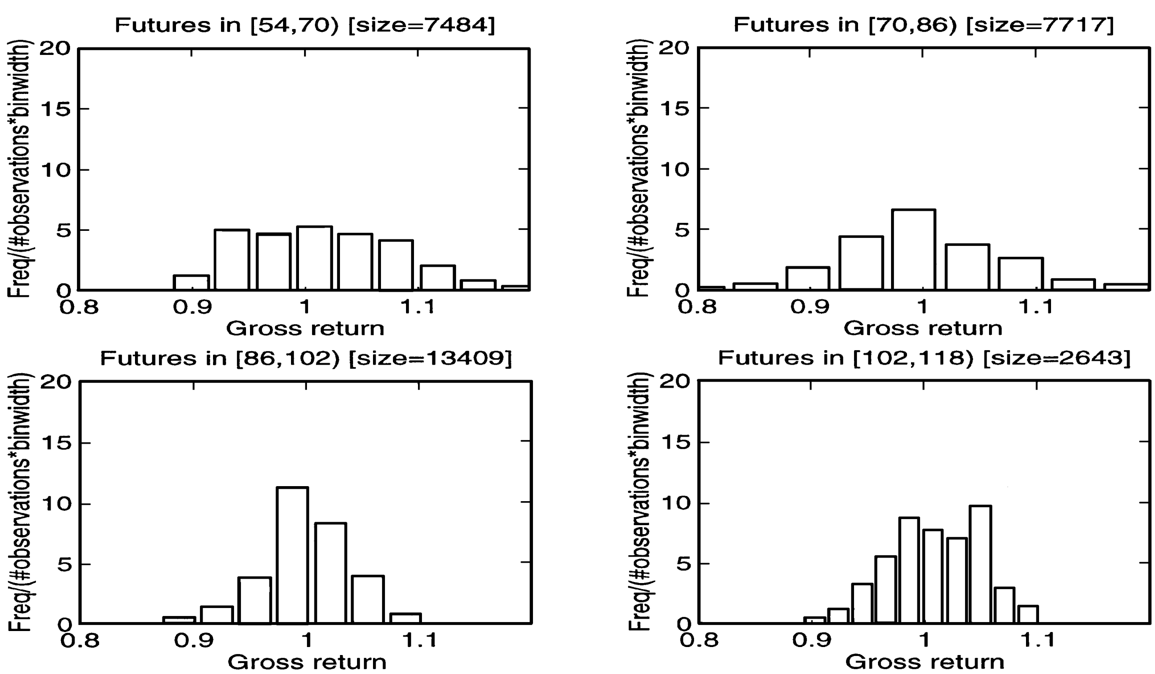


EXHIBIT 4
Normalized Histogram for Three Months' Return Data

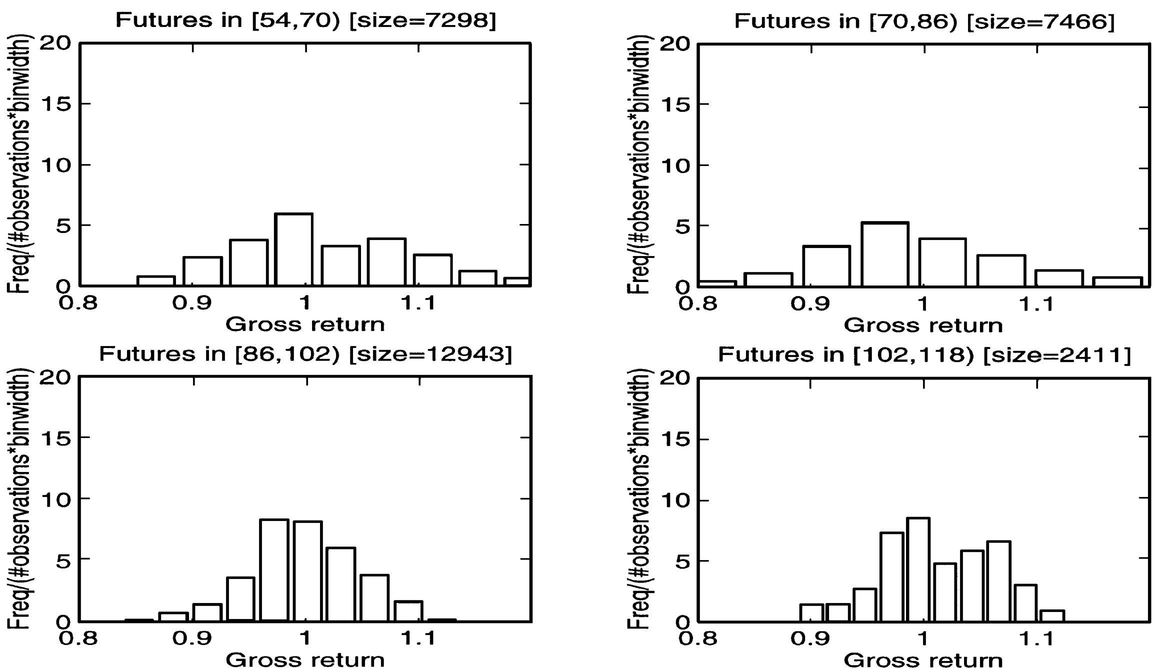
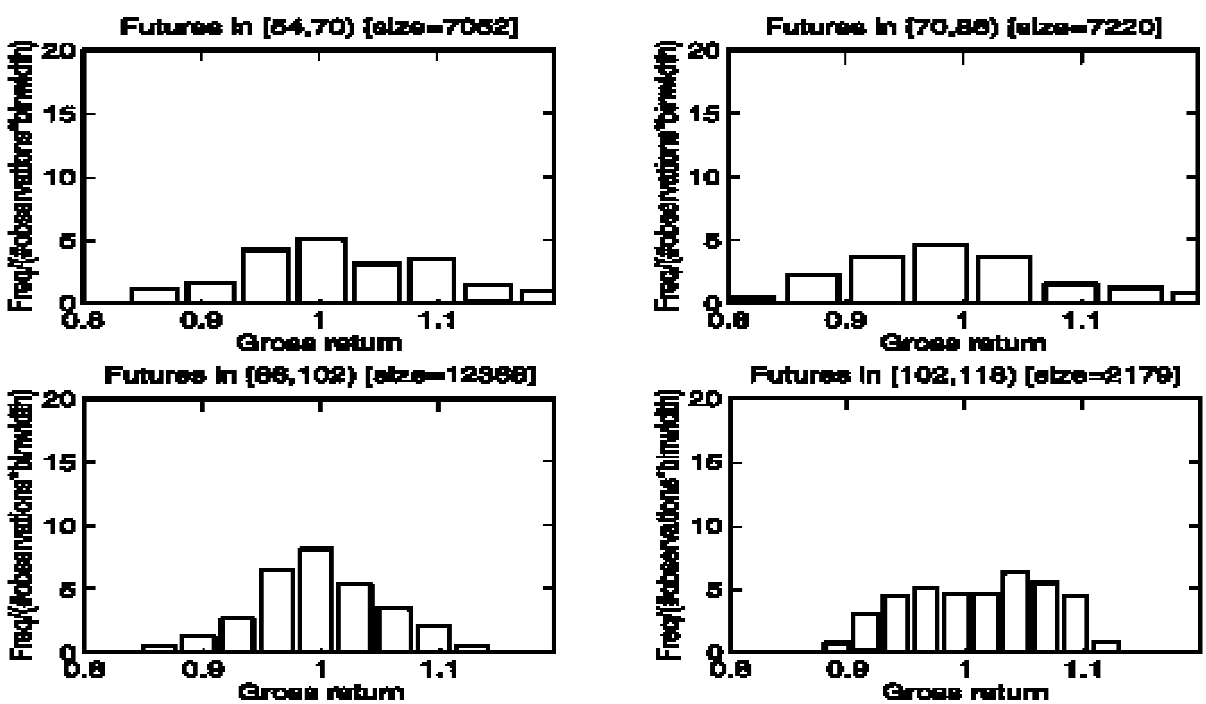


EXHIBIT 5
Normalized Histogram for Four Months' Return Data



mum entropy principle of distribution estimation.[5]

Specifically, for each combination of T and F, the convex problem is solved:

$$\hat{\pi} = \arg \max_{\sum_h \pi(h)=1} - \sum_{h=1}^{H} \pi(h) \log \pi(h) \qquad (2)$$

subject to $\sum_h \pi(h) P_h(T, F) = F$.

The constraint in (2) is a martingale constraint, requiring that the current futures price F equal the risk-neutral probability $\pi(h)$ weighted average (expectation) of the T-ahead futures prices $P_h(T, F)$. The maximand in (2) is the Shannon entropy of the risk-neutral distribution. As shown in Stutzer [1996], the risk-neutral probabilities solving (2) are an easily computed generalized exponential distribution called the *canonical* probability distribution denoted $\hat{\pi}(h)$, h = 1, …, H, in what follows.[6]

The canonical model value of a call option expiring at time T ahead with exercise price X is then easily calculated to be:

$$\text{Call} = \sum_{h=1}^{H} \hat{\pi}(h) \max[P_h(T, F) - X, 0] e^{-rT} \qquad (3)$$

where T denotes the time to expiration in years, and r denotes an estimate of the riskless short interest rate useful for this purpose. Puts could be valued by reversing X and $P_h(T, F)$ in (3).

IV. QUALITATIVE COMPARISON OF BLACK AND CANONICAL MODEL VALUES

To facilitate a qualitative comparison of Black and canonical model values, we choose a value of r = 5% for the continuously compounded short interest (i.e., discount) rate, and then convert each canonical value (3) to the volatility σ needed to make the Black model value (1) equal to that canonical model value.[7] We dub this the *canonical volatility,* in contrast to the usual Black model *implied* volatility, which is the volatility that makes the Black model value equal to the *market* price.

The canonical volatility skews reported in Exhibit 6 should predict historical patterns of Black model implied volatilities, because the latter are merely

EXHIBIT 6

Canonical Volatilities: Qualitative Predictions of Black Implied Volatility Skews

	Exercise Price (X)/F					
F	**0.95**	**0.97**	**0.99**	**1.01**	**1.03**	**1.05**
One-Month						
62	0.131	0.139	0.144	0.145	0.146	0.149
94	0.116	0.103	0.094	0.094	0.096	0.105
110	0.10	0.10	0.102	0.095	0.089	0.085
Four Months						
62	0.136	0.137	0.138	0.142	0.146	0.148
94	0.095	0.092	0.092	0.091	0.093	0.094
110	0.099	0.106	0.109	0.109	0.106	0.099

another way of quoting previously reported prices. Examination of Exhibit 6 quickly reveals that the canonical volatilities are much higher when the underlying futures price F is atypically low (around 62), i.e., the long-term bond yield is atypically high. This is consistent with empirical evidence showing a direct connection between the volatility and level of yields (see Sundaresan [1997, p. 50]).

In the more typical range of futures prices, we see that one-month in-the-money calls (X/F < 1) generally have higher canonical volatilities than other calls. Comparing one-month options to four-month options, we see that the model predicts that this pattern should be less significant for the four-month options. Both these canonical model predictions are consistent with the conclusions of Cakici, Chatterjee, and Wolf [1993].

V. OUT-OF-SAMPLE COMPARISON OF BLACK AND CANONICAL MODEL VALUES WITH MARKET PRICES

To compare the pricing performances, we choose closing market prices of CBOT bond futures options on twenty-one randomly selected days from August 1996 through January 1997.[8] On each day, we select the nearest-to-the-money option and those ±2 exercise prices from it, with between 1 and 4 months to expiration. In total, there are 105 of these out-of-sample options.[9]

In making the comparison between canonical and Black model prices, it is important to remember that *the canonical model (2) and (3) does not use any option market prices to predict other option market prices*. It is therefore in the spirit of the original Black-Scholes model, which is a theory that predicts option prices *solely* from information about the underlying asset volatility and the riskless discount rate. Subsequently, researchers have found that using an option-implied volatility parameter from, say, an at-the-money option improves the model's ability to predict the prices of other options.

Thus, in order to ensure a comparison of two theories that make use *solely* of information about the underlying asset and discount rate, we cannot use the Black model with an option-implied volatility. Instead, on each day selected, the Black model is calibrated with the *historical* volatility of the underlying futures price between June 1996 and that day.[10]

Exhibit 7 is a scatterplot of the Black model's dollar pricing errors versus degree of moneyness X/F; Exhibit 8 is the same scatterplot for the canonical model. Focusing on the main concentration of points in Exhibit 7, we see that there is an upward-sloping trend to the Black model pricing errors, the most negative occurring for low values of X/F. That is, the Black model has a tendency to relatively and absolutely underprice in-the-money calls. This pattern of mispricing is consistent with the findings of Cakici, Chatterjee, and Wolf [1993], who show that the relative mispricing of in-the-money calls persists even when using the Black model with an *option-implied* volatility, and *after* adding a value to it for the ability to exercise early.

Note in the canonical model's pricing error scatterplot in Exhibit 8 that the model does not have the tendency to underprice the in-the-money calls. That is, the errors do *not* have an obvious upward trend, and are centered closer to zero for the low values of X/F.[11]

Overall, the mean absolute value of the pricing errors (MAE) of the Black model is 11.9 cents, while the canonical model MAE is only 7.7 cents.[12] So both models seem to work well on the randomly sampled days during the out-of-sample period examined (August 1996-January 1997), with the canonical model outperforming the Black model on that basis.

We also compute the mean absolute percentage error (MAPE) in the *time* value of both models, i.e., the average value of the absolute percentage error (absolute pricing error of each option divided by the time value *above* the intrinsic value). This is a more stringent performance statistic, because it looks at the pricing error relative to the part of the option value that is hard to price (the time value above the intrinsic value). Both models have a time value MAPE of $13.6%.

EXHIBIT 7
Historical Volatility-Based Black Model Pricing Errors

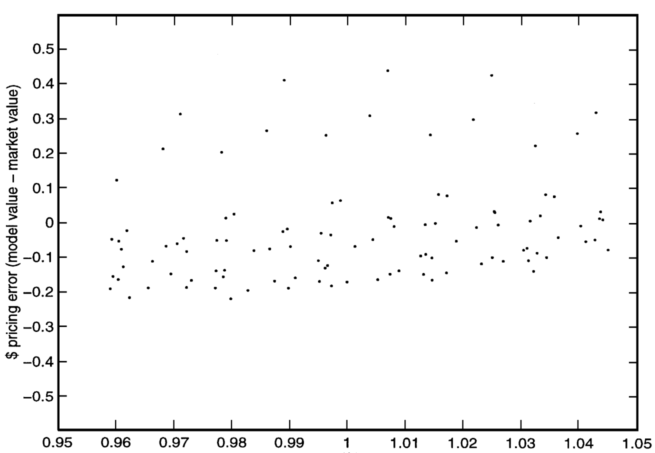
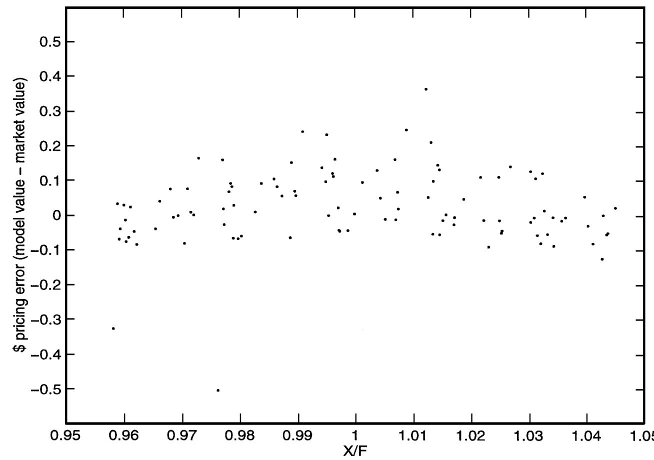


EXHIBIT 8
Canonical Model Pricing Errors



VI. OTHER MODELING ISSUES

A possible criticism of the Black model applied to fixed-income futures options is the assumption that the short interest rate is a known constant through the expiration date, while longer-term rates (and their associated note and bond futures prices) are assumed to be uncertain. As noted by Whaley [1986], short- and long-term interest rate volatility are to some degree separable.

In light of this, one should consider the fact that the most popular alternatives to the Black model (such as the numerous variants of the Ho and Lee [1986] model) each depend entirely on an ad hoc dynamic assumption about the movements of the short interest rate, which in turn severely constrains their comovements with the long-term interest rates most relevant for bond futures price movements.

While two-factor models provide a potential way around this, the parametric forms (such as Longstaff and Schwartz [1992]) still make it difficult to incorporate all the information in the historical series of bond futures prices.

Although there is no evidence that stochastic short-term rates are the reason for the Black model's empirical difficulties, the canonical model could be modified to incorporate stochastic short-term rates. Because the canonical model seems to perform adequately in the tests reported here, this modification was not made.

Another possible criticism of both the Black and the canonical model for pricing the CBOT note and bond futures options is the absence of any upward adjustment for the value of early exercise opportunities. Ramaswamy and Sundaresan [1985] suggest that the value added by the possibility of early exercise is "rather small, especially for options that are at-the-money" (p. 1327), and numerical simulations by Whaley [1986] indicate a significant effect only on the price of deep-in-the-money calls. Consistent with this, the seminal study by Cakici, Chatterjee, and Wolf [1993] on CBOT note and bond futures options prices makes the upward adjustment for early exercise, by adding the Barone-Adesi and Whaley [1987] calculated early exercise values to the Black model values, but finds that the results are "similar to" the model without this addition. It thus does not appear that failure to incorporate early exercise values explains the difference in the Black and canonical model pricing performances.

Finally, studies of actual early exercise of these options show that the vast majority occur with less than one month to expiration (Overdahl [1988]), and that some market participants have sometimes failed to exercise these options when it was rational to do so — even at expiration (see Gay, Kolb, and Yung [1989]). Considering all this, to be on the safe side, we made our pricing comparisons on options with more than one month to expiration that are not either too deeply in or out of the money.

Another possible criticism is the absence of any explicit adjustments to the estimated futures price distributions caused by the embedded options in the delivery process for the underlying bond futures contract. As discussed in Koenigsberg [1991], the seller has four options: what bond to deliver, when to deliver in the delivery month, and the "last-week" and "afternoon" wild card options. Ritchken and Sankarasubramanian [1995] report that the choice of delivery bond "is viewed as having the greatest effect." Their simulations suggest that this option's effect reduces the bond futures price by "about 2 percent" [1995, p. 275], with a maximum effect of 3.5% calculated by Koenigsberg [1991] using a different model.

Thus, it does not appear that the volatility skew's persistence over appreciable periods of time could be caused by the failure to explicitly adjust for these delivery options, although users concerned with this could modify their assessment of the futures price distribution to incorporate their views about the impact of these events.

VII. CONCLUSIONS AND FUTURE DIRECTIONS

The most comprehensive empirical valuation study of CBOT bond futures options discovered that the popular Black model of futures options is subject to a pattern of moneyness bias that is qualitatively similar to the extensively documented moneyness bias of the (Black-Scholes) model of stock index options.

To further investigate whether these moneyness biases can be attributed to misspecification of the model for the underlying price distribution, we modify the canonical model of Stutzer [1996] to value futures options, and apply it to the CBOT bond futures options. The model does not require assumptions implying a specific parametric form for the underlying futures price distribution.

Viewing canonical model values as predictors of option market prices, i.e., as predictors of implied volatilities, the model predicts that in the typical range of futures prices, short-term in-the-money calls should have higher implied volatilities than other calls, and that the implied volatilities will be higher when the current futures price is atypically high, i.e., when long-term bond rates are atypically high. These predictions are consistent with existing empirical evidence.

The out-of-sample market pricing performance of options with one to four months to expiration is examined during the last half of 1996, a period of relatively high futures prices (relatively low bond yields). Corroborating the earlier empirical evidence, we still observe a tendency for the Black model to underprice in-the-money calls relative to others; that is, the implied volatilities are inversely related to the exercise price. The canonical model does not exhibit this mispricing pattern during the testing period.

The canonical model outperforms the historical volatility-based Black model during this period, with a mean absolute pricing error of 7.7 cents versus 11.9 cents for the Black model. Both models, however, have the same mean absolute *percentage* errors in the *time* value of the options.

The results would seem to warrant further development in several directions. Of course, our unconstrained model could be applied to other interest rate, equity, or commodity futures options. In addition, one could further constrain the model, along the lines suggested in Stutzer [1996], in order to improve its pricing performance, especially for options with longer terms to expiration. Finally, it should be possible to incorporate the additional complexity of a stochastic short-term rate, or to assign a value for the possibility of early exercise by using the difference between another model value (say, Black) and the higher value computed from its modification for early exercise (say, the Barone-Adesi and Whaley [1987] modification), in applications where these are believed to be major sources of pricing error. This research was supported by a grant from the Chicago Board of Trade Educational Research Foundation.

ENDNOTES

This research was supported by a grant from the Chicago Board of Trade Educational Research Foundation.

[1]Cakici, Chatterjee, and Wolf also use the Barone-Adesi and Whaley [1987] method to incorporate the additional value of the early exercise feature. They conclude that "results for the European pricing model are similar to the American model" [1993, p. 7]. Hence, failure to incorporate early exercise here is not a serious problem.

[2]We use histograms to form our estimates of the actual distribution at option expiration, but users could substitute their own assessments if desired.

[3]This is the simplest imaginable form of kernel smoothing, using a uniform kernel with a bandwidth of \$8. The \$8 bandwidth was found to achieve smaller pricing errors than smaller bandwidths, yet still be computationally tractable.

[4]If the underlying asset price is a spot rather than a forward or futures, one would also have to discount the T-ahead prices.

[5]Other uses of the maximum entropy principle in option pricing are in Stutzer [1994], Hawkins, Rubinstein, and Daniell [1996], and Buchen and Kelly [1996], and some related uses in asset pricing are developed in Stutzer [1995] and in Kitamura and Stutzer [1997].

[6]The canonical option model was named in order to honor Gibbs [1902], who was America's first internationally renowned scientist.

[7]Different values of the discount rate r will not change the nature of our conclusions.

[8]Because the bond futures and bond futures options markets close at the same time, there is less chance of nonsimultaneous quotes changing the nature of conclusions concerning the *relative* pricing performance of the two models.

[9]On January 21, 1997, there were two options with different expiration dates that met these criteria.

[10]One can easily modify the canonical risk-neutral probabilities to reflect any actual option market prices, analogous to the use of an option-implied volatility parameter. One adds an additional constraint to (2) requiring that each chosen option market price equal the discounted risk-neutral expected value of that option's payoff at expiration. Because the canonical model performs well with just the single (martingale) constraint in (2), we did not see any need to use option market prices in this fashion. After all, if the goal were just to predict some option prices from knowledge of other option prices, completely atheoretical non-parametric methods, like the neural network method in Hutchinson, Lo, and Poggio [1994] might work better than any theory predicated on the absence of arbitrage opportunities involving trading in the underlying and riskless assets.

[11]The three outliers in Exhibit 8 all occur on October 8, 1996, and are for options with twenty-six days to expiration.

[12]The Black model's performance was adversely influenced by outliers occurring on three days in 1996 —August 5, August 9, and August 19 — all for options with times to expiration of three months or more.

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